Material: Ch 3 Fourier Series

Ch 4. Wave equ. (1-dim)

up to 4.4

Fourier series:

3 types: cosine series, Sine series } for functions $f: [0,L] \to \mathbb{R}$

full Fourier series for functions $f: [-L,L] \to \mathbb{R}$

If $f: [-L,L] \to \mathbb{R}$ odd $\Rightarrow$ its Fourier series = sine series for $f|_{[0,L]}$

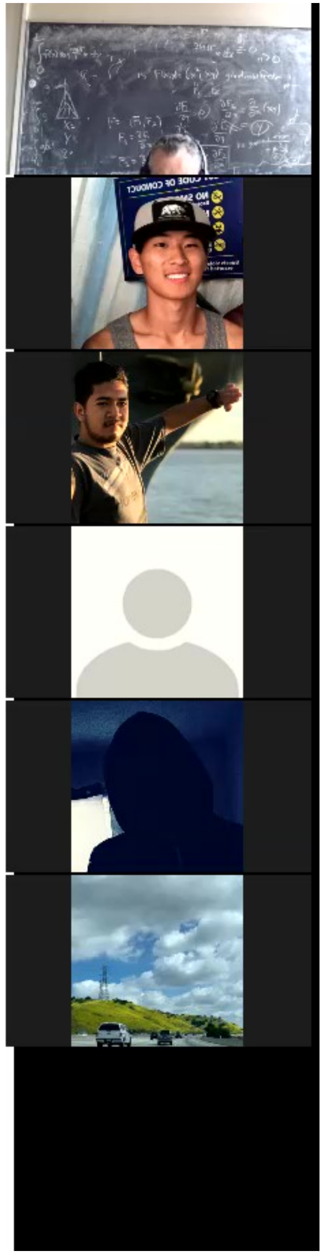
" " even " " Cosine series " "

basic assumption on f: piecewise smooth.

- continuous except for finitely many jump discontinuities

Fourier's Theorem: (applies for all 3 types):

$$\text{value of Fourier series at } x = \begin{cases} f(x) & \text{if } f \text{ cont. at } x \\ = \frac{1}{2}\left[f(x+) + f(x-)\right] & \text{if } f \text{ jumps at } x \\ \text{or } \frac{1}{2}\left[f(-L+) + f(L-)\right] & \text{if } x = \pm L \text{ (for full series)} \end{cases}$$



Formulas: e.g. for cosine series

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx$$

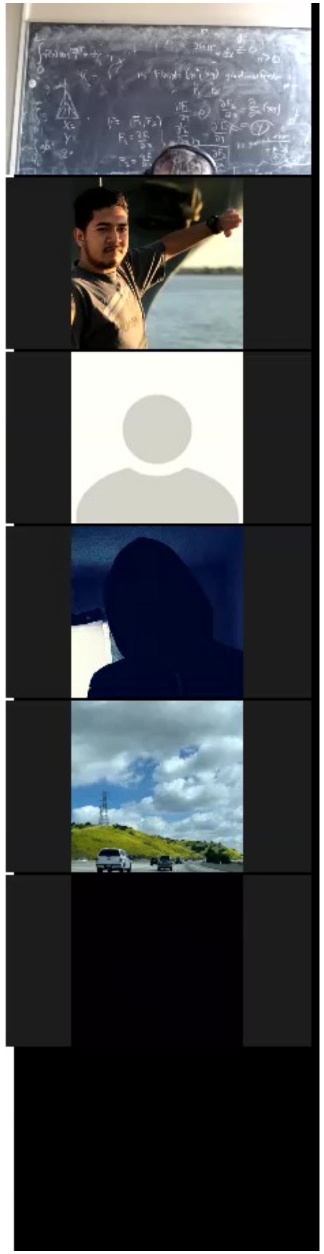
Connection between Fourier/cos/sin series of f and of f'

Theorem: Assume both f and f' are piecewise smooth
$f: [-L, L] \to \mathbb{R}$

If Fourier series of f is continuous $\Rightarrow$
Fourier series of f' obtained via differentiating Fourier series of f term by term.

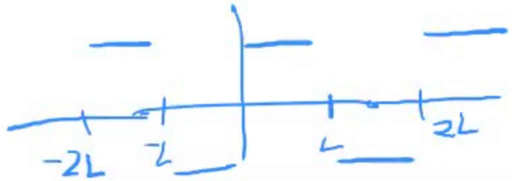
Examples: ⓐ even extension of $f: [0, L] \to \mathbb{R}$
$f(x) = x$

contin. $\Rightarrow$ theorem applies



Fourier series for f $\quad \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^2} \cos \frac{n\pi x}{L}$

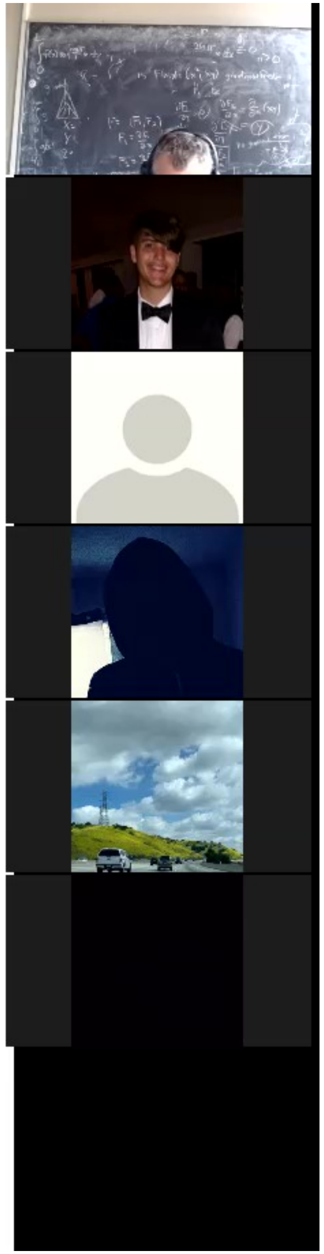
Fourier series for f' $=$



just differentiate the one for f

$$= \quad - \frac{4\cancel{L}}{\pi^{\cancel{2}}} \sum \frac{1}{n^{\cancel{2}}} \left(- \sin \frac{n\pi}{L} x\right) \frac{\cancel{n\pi}}{\cancel{L}}$$

$$= \quad \sum \frac{4}{n\pi} \sin \frac{n\pi}{L} x$$



Remark: If $A_0 = 0$ in cosine series
(or $a_0 = 0$ in full Fourier series)

and $F' = f$

$\Rightarrow$ get Fourier series of F by integrating
the one for f (up to a constant)

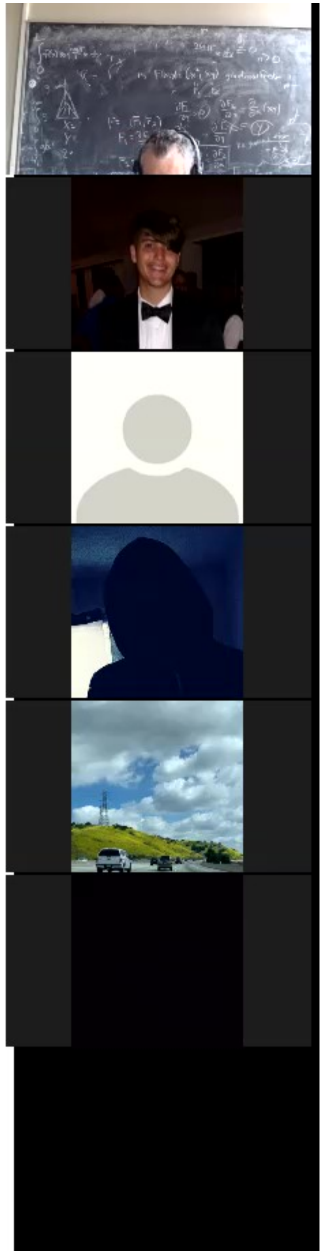
Example: sine series for $f(x) = x$ given by

$$2 \sum \frac{L}{n\pi} (-1)^{n+1} \sin \frac{n\pi x}{L}$$

$\Rightarrow$ get cosine series
for $\frac{x^2}{2} = F(x)$ by integrating

$$= 2 \sum \frac{L}{n\pi} (-1)^{n+1} \left(-\cos \frac{n\pi x}{L}\right) \cdot \frac{L}{n\pi}$$

$$= 2 \sum \frac{L^2}{(n\pi)^2} (-1)^n \cos \frac{n\pi x}{L}$$



$$\text{const} = A_0 = \frac{1}{2L}\int_{-L}^{L} \frac{x^2}{2}\,dx$$

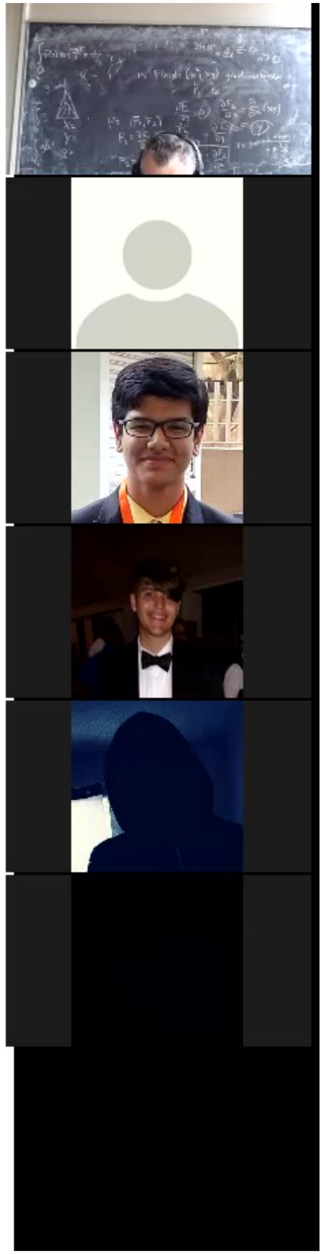
$$= \frac{1}{L}\int_{0}^{L} \frac{x^2}{2}\,dx$$

Wave Equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + Q$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}$$

$$c^2 = \frac{T}{\rho}$$



usual way to solve:

here e.g. for $$\rho \frac{\partial^2 u}{\partial t^2} = T \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}$$

$$u(x,t) = \phi(x)\, h(t)$$

plug into PDE

$$\rho\, \phi(x) h''(t) = T \phi''(x) h(t) - \beta \phi(x) h'(x)$$

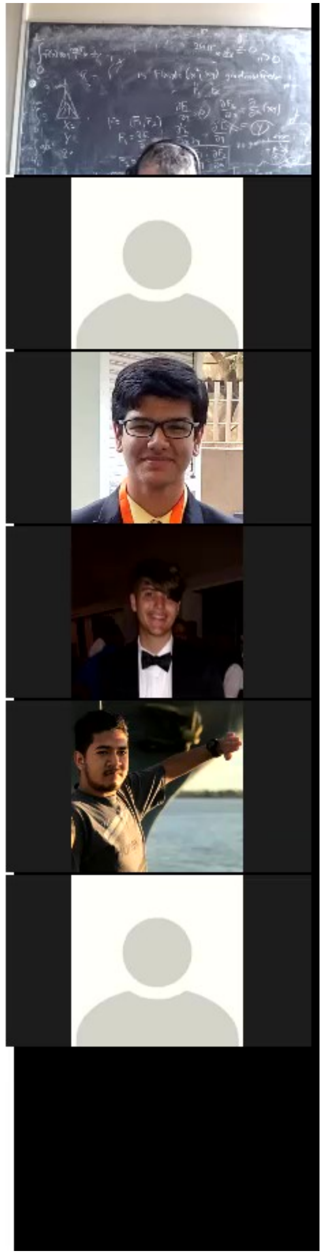
$$\rho\, \phi(x) h''(t) + \beta \phi(x) h'(t) = T \phi''(x) h(t) \quad \Big|\ \frac{1}{\phi(x) h(t)}$$

$$\frac{h''(t)}{h(t)} + \frac{\beta}{\rho} \frac{h'(t)}{h(t)} = \frac{T}{\rho} \frac{\phi''(x)}{\phi(x)} = -\lambda \, T/\rho \quad \Big|\ \frac{1}{\rho}$$

$$\frac{T}{\rho} = c^2$$

$\Rightarrow$ get 2 ODE's:

$$\boxed{\begin{aligned} \phi''(x) &= -\lambda \phi(x) \\ h''(t) + \beta/\rho\, h'(t) &= -\lambda \underbrace{T/\rho}_{= c^2} h(t) \end{aligned}}$$



use boundary conditions.

here: fixed boundaries $u(0,t)=0=u(L,t)$

$$\Rightarrow \quad \phi(0)=0=\phi(L)$$

$$\Rightarrow \quad \boxed{\phi(x) = \sin \frac{n\pi}{L} x \quad , n=1,2\ldots}$$

$$\boxed{\lambda = \frac{n^2\pi^2}{L^2}}$$

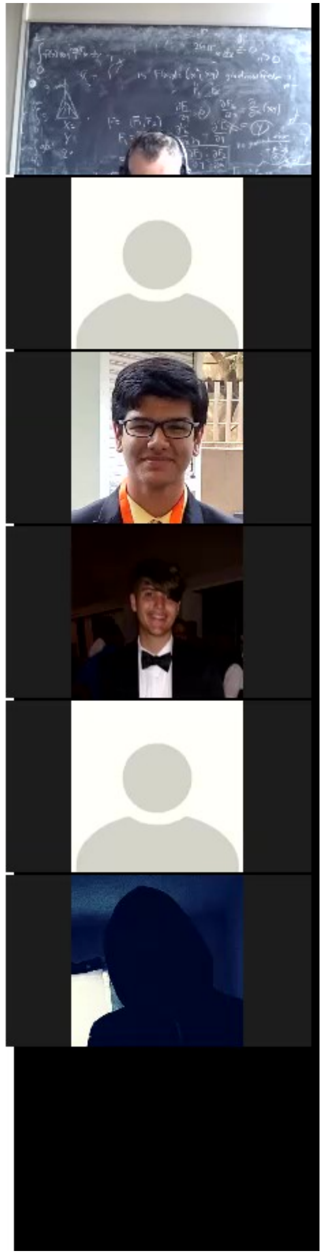
plug into other ODE:

$$h''(t) + \frac{\beta}{\rho} h'(t) + \frac{\lambda T}{\rho} h(t) = 0$$

$\Rightarrow$ char. equ.

$$x^2 + \frac{\beta}{\rho} x + \frac{\lambda T}{\rho} = 0$$

$$\Rightarrow \quad x_{1,2} = \frac{1}{2}\left(-\beta/\rho \pm \sqrt{\beta^2/\rho^2 - 4\lambda T/\rho} \right)$$



$\Rightarrow$ get complex solutions

Theorem: if char. equation has complex solutions $a \pm ib$

$\Rightarrow$ ODE has solutions

$$e^{at}(A\cos bt + B\sin bt)$$

$\Rightarrow$ get eigenfunctions

$$\phi(x)\,h(t) = \sin\frac{n\pi}{L}x\; e^{-\beta t/2\rho}\left(A_n\cos\sqrt{4n^2\pi^2T\rho/L^2-\beta^2}\,t + B_n\sin(\quad)\right)$$

$n=1,2,\dots$